Stable spinning optical solitons in three dimensions

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We introduce spatiotemporal spinning solitons (vortex tori) of the three-dimensional nonlinear Schrödinger equation with focusing cubic and defocusing quintic nonlinearities. The first ever found completely stable spatiotemporal vortex solitons are demonstrated. A general conclusion is that stable spinning solitons are possible as a result of competition between focusing and defocusing nonlinearities.

Optical solitons (spatial, temporal, or spatiotemporal) are self-trapped light beams or pulses that are supported by a balance between diffraction and/or dispersion and various nonlinearities. They are ubiquitous objects in optical media [1]. Spatiotemporal solitons (STS) [2], alias superspikes [3, 4] or light bullets [5], were found in many works [2] - [9]. Although they cannot be stable in the uniform Kerr ($\chi^{(3)}$) medium [6], stability can be achieved in saturable [3, 7], quadratically nonlinear ($\chi^{(2)}$) [2, 8], and graded-index Kerr media [9]. While a fully localized STS in three dimensions (3D) has not yet been found in an experiment, 2D ones were observed in a bulk $\chi^{(2)}$ medium [10]. The interplay of spatio-temporal coupling and nonlinearity may also play an important role in self-defocusing media [11].

Spinning (vortex) solitons are also possible in optical media. Starting with the works [12], both delocalized ("dark") and localized ("bright") optical vortices in 2D were investigated [13, 14, 15]. In the 3D case they take the shape of a torus ("doughnut") [16, 17]. However, the only previously known physical model which could support *stable* 3D vortex solitons is the Skyrme model [18], which has recently found a new important application to Bose-Einstein condensates (BEC) [19]. Our objective in this paper is to identify fundamental models of the nonlinear-Schrödinger (NLS) type in 3D that give rise to stable spinning solitons, as NLS models are much simpler and closer to more experimental situations, having applications to optics, BEC, plasmas, etc. (see below).

For bright vortex solitons stability is a major issue as, unlike their zero-spin counterparts, the spinning solitons are prone to destabilization by azimuthal perturbations. In 2D models with $\chi^{(2)}$ and saturable nonlinearities an azimuthal instability was revealed by simulations [14] and observed experimentally [15]. As a result, a soliton with spin 1 splits into two or three fragments, each being a moving zero-spin soliton. Simulations of the 3D spinning STS in the $\chi^{(2)}$ model also demonstrates its instability-induced splitting into separating zero-spin solitons [17]. Nevertheless, the $\chi^{(2)}$ nonlinearity acting in combination with the self-*defocusing* Kerr ($\chi^{(3)}$) nonlinearity, gives rise to the first examples of stable spinning (ring-shaped) 2D solitons with spin $s = 1$ and 2 [20]. It should be stressed that all the 2D spinning solitons actually represent static spatial beams; on the contrary, 3D solitons are moving spatio*temporal* ones, which are localized not only in the transverse plane, but also in the propagation coordinate, see below.

A model which may support stable spinning solitons in 3D is the one with a cubic-quintic (CQ) nonlinearity, which (in terms of optics) assumes a nonlinear correction to the medium's refractive index in the form $\delta n = n_2 I - n_4 I^2$, I being the light intensity. The CQ nonlinearity was derived, starting from the Maxwell-Bloch equations, for light propagation combining resonant interaction with two-level atoms and dipole interactions between the atoms [21],

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or the Kerr nonlinearity of a waveguide [22] (see also Ref. [23]). A unifying feature of those media is competition between different nonlinearities, and stable solitons may exist in the range of intensities where the competition takes place (which may be controlled, for instance, through the density of two-level atoms in a $\chi^{(3)}$ waveguide). In fact, the NLS equation of the CQ type is a generic model, which also applies to Langmuir waves in plasmas[24] and BEC [25] (although in the latter case, three-body interactions, which give rise to the quintic term, may induce losses through recombination of the colliding atoms, thus making the quintic coefficient complex).

In the first simulations of 2D solitons with spin 1 in the CQ model, it was found that they propagated in a stable way, provided that their energy is not too small [26]. A later analysis, based on the computation of linear-stability eigenvalues, demonstrated that some of the spinning 2D solitons considered in Ref. [26] are subject to a weak azimuthal instability. Nonetheless, in another part of their existence region, with very large energies, solitons with spin $s = 1$ and $s = 2$ were confirmed to be truly stable in the 2D CQ model [27] (all the solitons with $s \geq 3$ are unstable).

It was recently shown by direct simulations of the CQ model [28] that 3D spinning solitons with moderate energies are unstable against azimuthal perturbations, while the ones with very large energies, i.e., broad "doughnuts" with a small hole in the center, were robust under propagation. However, a consistent stability analysis makes it necessary to compute eigenvalues of small perturbations. We will conclude that sufficiently broad STS with spin $s = 1$ are stable, the stability region occupying $\approx 20\%$ of their existence region, while all the STS with $s \geq 2$ are unstable.

The evolution of the electromagnetic field envelope A in the dispersive CQ medium is governed by the NLS equation,

$$2i\kappa_0 A_z + \nabla_\perp^2 A + \kappa_0 D A_{\tau\tau} + 2\kappa_0^2 (n_2/n_0)|A|^2 A - 2\kappa_0^2 (n_4/n_0)|A|^4 A = 0,\ \tau \equiv t - z/V, \tag{1}$$

where z and t are the propagation coordinate and time, κ_0 and V are the propagation constant and group velocity of the carrier wave, $D > 0$ is the temporal dispersion, $\nabla_\perp^2$ acting on the transverse coordinates x and y. Equation (1) does not include higher-order effects, such as self-steepening, stimulated Raman scattering, non-paraxial diffraction, and third-order temporal dispersion (which, in another context, were taken into regard for spatiotemporal superspikes in Refs. [4]), as we anticipate that only broad solitons (with the temporal width ~ 1 ps), for which these effects are small, may be stable.

Defining rescaled variables $u = \sqrt{n_4/n_2}A$, $T = n_2\sqrt{2\kappa_0/Dn_0n_4}\tau$, $Z = \left(\kappa_0 n_2^2/n_0 n_4\right) z$, and $(X,Y) = \kappa_0 n_2 \sqrt{2/n_0 n_4}\,(x,y)$, we transform Eq. (1) into a normalized form [28],

$$iu_Z + (u_{XX} + u_{YY} + u_{TT}) + |u|^2 u - |u|^4 u = 0. \tag{2}$$

STS solutions to Eq. (2) are sought as $u = U(r,T)\exp(is\theta)\exp(i\kappa Z)$, where r and θ are the polar coordinates in the transverse plane, κ is a propagation constant parameterizing the family of solutions sought for, and s is an integer spin. The real amplitude U obeys the equation

$$\left(U_{rr} + r^{-1}U_r - s^2 r^{-2} U + U_{TT}\right) - \kappa U + U^3 - U^5 = 0, \tag{3}$$

supplemented by the condition that U must decay exponentially as $r \to \infty$ and $T \to \infty$ (due to the definition of τ in Eq. (1), the latter condition implies that the solution's snapshot taken at $t = \text{const}$ is localized in the propagation coordinate z - in fact, exactly the same way as 1D solitons are localized in optical fibers [29]).

Equation (2) conserves the energy $E = \int\int\int |u(X,Y,T)|^2\, dXdYdT$, Hamiltonian

$$H = \int\int\int \left[|u_X|^2 + |u_Y|^2 + |u_T|^2 - (1/2)|u|^4 + (1/3)|u|^6\right] dXdYdT, \tag{4}$$

momentum (equal to zero for the solutions considered), and angular momentum in the transverse plane, $L = \int\int\int (\partial\phi/\partial\theta)\,|u|^2 dXdYdT$, where ϕ is the phase of the complex field u. Relations between L, H and E for a stationary spinning soliton follow from Eq. (3): $L = sE$; $H = \kappa E - \frac{2}{3}\int\int 2\pi r U^6(r,T)drdT$ [28].

We have numerically found families of 3D spinning solitons with a toroidal shape. To quantify the solutions, in Fig. 1 we show the propagation constant κ and H vs. E for both $s = 0$ and $s = 1,2$ solitons. They exist for E exceeding a threshold value, which increases with s. The full and dashed lines in Fig. 1 correspond to stable and unstable branches according to results presented below. The $s = 0$ branch of the solutions is divided into stable and unstable portions on the basis of the known criterion which states that the fundamental ($s = 0$) soliton branch undergoes a change in the stability where $dE/d\kappa = 0$ [30]. However, this criterion ignores azimuthal instability, which is frequently fatal for spinning solitons.

The most revealing information on the stability of solitons is provided by analysis of a linearized version of Eq. (2). To this end, we seek perturbation eigenmodes of the general form,

$$\begin{aligned} u(Z,r,T,\theta) - U(r,T)\exp\left[i(s\theta + \kappa Z)\right] \;=\; & f(r,T)\exp\left\{\lambda_n Z + i[(s+n)\theta + \kappa Z]\right\} \\ & + g^*(r,T)\exp\left\{\lambda_n^* Z + i[(s-n)\theta + \kappa Z]\right\}, \end{aligned} \tag{5}$$

where $n>0$ is an arbitrary integer azimuthal index of the perturbation, λ_n is the (complex) instability growth rate sought for, and the functions f and g obey equations

$$\begin{aligned} i\lambda_n f+\frac{\partial^2 f}{\partial T^2}+\frac{\partial^2 f}{\partial r^2}+r^{-1}\frac{\partial f}{\partial r}-(s+n)^2r^{-2}f-\kappa f+\left(2-3U^2\right)U^2 f+\left(1-2U^2\right)U^2 g &= 0, \\ -i\lambda_n g+\frac{\partial^2 g}{\partial T^2}+\frac{\partial^2 g}{\partial r^2}+r^{-1}\frac{\partial g}{\partial r}-(s-n)^2r^{-2}g-\kappa g+\left(2-3U^2\right)U^2 g+\left(1-2U^2\right)U^2 f &= 0. \end{aligned} \tag{6}$$

The solutions must decay exponentially at $r\to\infty$, and vanish as $r^{|s\pm n|}$ at $r\to 0$.

To solve Eqs. (6), we used a known numerical procedure [14, 31], which produces results presented in Fig. 2. The most persistent unstable eigenmode has $n=2$, for both $s=1$ and $s=2$. As is seen in Fig. 2, with the increase of κ, the instability of the soliton with $s=1$, accounted for by $\mathrm{Re}\,\lambda_2$, disappears at $\kappa=\kappa_{\mathrm{st}}\approx 0.13$, and the stability region extends up to $\kappa=\kappa_{\mathrm{offset}}^{(3\mathrm{D})}\approx 0.17$, corresponding to infinitely broad solitons (which implies that the vortex of the dark-soliton type [12], that may be regarded as an infinitely broad spinning soliton, is stable too). The relative width of the stability region is $\left(\kappa_{\mathrm{offset}}^{(3\mathrm{D})}-\kappa_{\mathrm{st}}\right)/\kappa_{\mathrm{offset}}^{(3\mathrm{D})}\approx 0.2$. However, there is *no* stability region for 3D solitons with $s=2$, in contrast to the 2D vortex solitons in the CQ model [27]. In the case when a spinning soliton is unstable, its instability is *oscillatory*; the corresponding frequency, $\mathrm{Im}\lambda$, is of the same order of magnitude as $\mathrm{Re}\lambda$ at the maximum-instability point (see Fig. 2), and λ becomes purely imaginary at $\kappa=\kappa_{\mathrm{st}}$.

The above results were checked in direct simulations of Eq. (2) by means of the Crank-Nicholson scheme combined with the Gauss-Seidel iteration procedure. In Fig. 3 we show the amplitude and energy vs. Z for the soliton with $s=1$, generated by two different initial configurations, with the same energy of the initial configuration, $E_0=13070$. Robustness of the spinning STS is attested to by the fact that it can be generated from a Gaussian with a nested vortex whose shape is far from the soliton's exact form. (Energy loss evident in Fig. 3(b) is caused by emission of radiation in the course of the formation of the stable STS; naturally, the loss is larger for the initial Gaussian configuration, which is farther from the exact soliton's shape.) Figure 4 shows the gray-scale contour plots of the intensity and phase of both the input Gaussian with a nested vortex and emerging spinning STS at $Z=400$.

The instability of the $s=2$ solitons is illustrated in Figs. 5 and 6. The azimuthal instability breaks them into zero-spin solitons which fly out tangentially, relative to the circular crest of the original soliton. It is noteworthy that, at an early stage of the evolution shown in Fig. 5, the spinning soliton splits into two fragments, in accordance with the growth-rate calculations predicting the dominant instability to be against the $n=2$ perturbation mode, but the subsequent nonlinear evolution is more involved. In Fig. 6(b), the spinning STS splits into three fragments with *unequal* energies. The fragmentation of this soliton (having $\kappa=0.09$) into three parts is in accordance with the growth-rate calculation, see Fig. 2(b).

In conclusion, we have found the first example of stable three-dimensional spinning solitons in a dispersive medium which combines cubic and quintic nonlinearities. Only sufficiently broad solitons with spin $s=1$ may be stable. However, the existence of stable spinning 3D solitons is a generic fact, as it is not limited to the cubic-quintic nonlinearity: our preliminary studies indicate that these stable physical objects may also occur in media with competing quadratic and self-defocusing cubic nonlinearities. In fact, a condition which turns out to be necessary for the existence of stable spinning solitons is a competition between two different nonlinearities, one focusing and the other one defocusing.

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FIG. 1: The propagation constant κ (a) and Hamiltonian H (b) of the 3D spinning soliton vs. its energy E.

FIG. 2: The growth rate of perturbations, Reλ, with different values of the azimuthal number n (indicated by labels near the curves) vs. the soliton's propagation constant κ: (a) $s = 1$; (b) $s = 2$.

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FIG. 3: Evolution of the maximum amplitude of the soliton with $s = 1$ (a) and its energy (b), as generated by initial configurations in the form of a Gaussian with a nested vortex (continuous curves) or a torus close to the stationary spinning soliton corresponding to $\kappa = 0.15$ (dashed curves).

FIG. 4: The formation of the soliton with spin $s = 1$: (a) the initial Gaussian with a nested vortex; (b) its phase field; (c) the spinning soliton at $Z = 400$; (d) the phase field at $Z = 400$. A cross section of the fields at $T = 0$ is shown.

FIG. 5: Gray-scale plots showing the developing instability of the spinning soliton with $s = 2$ and $\kappa = 0.13$ at $Z = 0$ (a), $Z = 600$ (b), $Z = 620$ (c), and $Z = 640$ (d).

FIG. 6: Isosurface plots illustrating the instability of the $s = 2$ soliton with $\kappa = 0.09$: (a) $Z = 0$, (b) $Z = 250$.